Journal of Structural Geology, Vol. 11, No. 5, pp. 625 to 627, 1989
Printed in Great Britain

0191-8141/89 $03.00 + 0.00

Brevia

SHORT NOTE

A construction for shear stress on a generally-oriented plane

W. D. Means

Department of Geological Sciences, State University of New York at Albany, 1400 Washington Avenue, Albany, NY 12222, U.S.A.

(*Received* 11 *November* 1988; *accepted* 16 *February* 1989)

Abstract—The direction and magnitude of the shear stress in a generally-oriented plane can be found graphically by decomposing the three-dimensional stress tensor into a hydrostatic component, a uniaxial compression and a uniaxial tension.

INTRODUCTION

Lisle (1989) has described a new stereographic method for finding the direction of shear stress on a generally-oriented plane in three dimensions, when the principal stresses are known. This note describes yet another method, about as simple as Lisle's, which yields the magnitudes and directions of the shear and normal stresses, and the sense of the shear stress. Lisle's method and the present one are both easier to use than earlier published methods (references in Lisle 1989). An unpublished construction by Etchecopar (1984), yields the direction of the shear stress and is also easy to use. It has several features in common with the present method, as discussed later.

PROCEDURE

Plot an *upper-hemisphere* projection showing the plane of interest, its pole P, and the σ_1 and σ_3 principal stress directions (Fig. 1a). An equal-angle, equal-area or orthographic projection may be used.

Rotate the plane to horizontal through its dip angle, and its pole to vertical, using the strike line of the plane as the axis of rotation. Rotate the σ_1 and σ_3 directions about the same axis, by the same angle (Fig. 1b).

Measure angles

$$a = \mathrm{P} \wedge \sigma_1$$
$$b = \mathrm{P} \wedge \sigma_3,$$

and calculate the quantities

$$\tau_1 = (\sigma_1 - \sigma_2)\cos a \sin a$$
$$\tau_3 = (\sigma_2 - \sigma_3)\cos b \sin b,$$

where σ_1, σ_2 and σ_3 are the magnitudes of the principal stresses (positive for compressions).

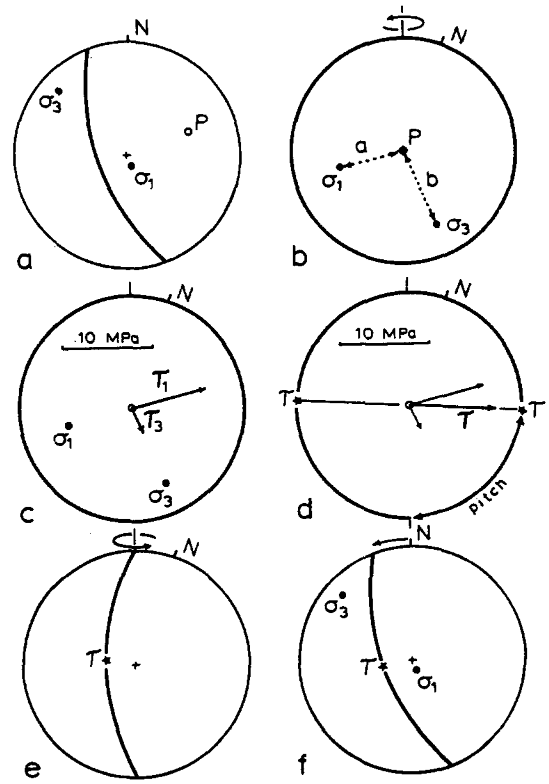

Fig. 1. Steps in construction. (a) Geographic orientations of plane of interest, its pole P, and the directions of σ_1 and σ_3. *Upper-hemisphere* stereographic (equal-angle) projection. The plane dips 60° towards azimuth 070°. σ_1 plunges 78° toward 337°; σ_3 plunges 11° toward 132°. (b) Projection after rotation of plane to horizontal about its strike direction. Angles *a* and *b* are 60° and 72°, respectively. (The position of the North mark on the paper is shown although the plane of projection no longer contains the N–S direction.) (c) Components τ_1 and τ_3 drawn to scale indicated, using 50, 30 and 20 MPa as the magnitudes of the principal stresses. (d) Vector addition of τ_1 and τ_3 to find τ, the total shear stress in the plane, and its points in projection τ. The pitch of τ is about 86°S. (e) and (f) Restoration of plane and τ point in it to geographic orientation. τ plunges at about 59° toward 077°, with magnitude of about 9.5 MPa, hangingwall-down.

Draw a vector of length τ_1 extending from P *away* from the σ_1 point, and a vector of length τ_3 extending from P *toward* the σ_3 point, using any convenient scale (Fig. 1c).

Add the resulting representations of τ_1 and τ_3 vectorially to obtain the total stress vector τ acting in the plane (Fig. 1d). Extend the τ arrow to intersect the primitive circle, to find the projection point representing the direction of τ in the plane, and read its pitch if desired (Fig. 1d).

Determine the magnitude of τ from its length. Obtain the sense of shear of hangingwall-relative-to-footwall by noting the direction in which the τ arrow points.

Recover the geographic orientation of τ and read its trend and plunge, by rotating the plane, with its included τ point, back to the original dip and strike of the plane (Figs. 1e & f). With practice, all these construction steps can all be carried out on one piece of tracing paper.

The normal stress on the plane acts in the direction of P and has magnitude

$$\sigma = (\sigma_1 - \sigma_2)\cos^2 a + (\sigma_3 - \sigma_2)\cos^2 b + \sigma_2.$$

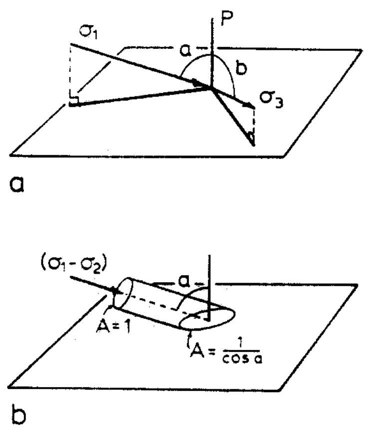

Fig. 2. (a) Diagram corresponding to Fig. 1(b), showing the plane of interest and the orthogonal projections (heavy lines) of σ_1 and σ_3 upon it, which become the directions of τ_1 and τ_3. (b) Uniaxial compression $(\sigma_1 - \sigma_2)$ acting in the σ_1 direction (arrow). The force delivered to the plane through a unit cylinder of material, of magnitude $(\sigma_1 - \sigma_2)$, becomes spread over an elliptical area of $1/\cos a$, giving rise to a stress in the σ_1 direction of $(\sigma_1 - \sigma_2)\cos 2a$. The component of this stress acting parallel to the plane is τ_1 and is obtained by multiplication by $\sin a$.

JUSTIFICATION

The method makes use of a three-way decomposition of the stress tensor into a hydrostatic component of magnitude σ_2, a uniaxial compression of magnitude $(\sigma_1 - \sigma_2)$ and a uniaxial tension of magnitude $(\sigma_3 - \sigma_2)$. These sum to give the total stress as follows (where axes x_i are chosen parallel to the corresponding principal stresses σ_i):

$$\underset{\text{hydrostatic stress}}{\begin{bmatrix} \sigma_2 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_2 \end{bmatrix}} + \underset{\text{uniaxial compression}}{\begin{bmatrix} (\sigma_1 - \sigma_2) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$$

$$+ \underset{\text{uniaxial tension}}{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (\sigma_3 - \sigma_2) \end{bmatrix}} = \underset{\text{total stress}}{\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}}.$$

Any state of stress can be decomposed into these three parts.

The shear stress on any plane arises only from the uniaxial compression and the uniaxial tension. These contribute components, here called τ_1 and τ_3, respectively, to the total shear stress acting on the plane. Because the uniaxial compression and tension are axisymmetric about the σ_1 and σ_3 directions, τ_1 and τ_3 must lie in the directions of the orthogonal projections of σ_1 and σ_3 upon the plane (Fig. 2a). The τ_1 arrow is drawn directed away from the σ_1 point in Fig. 1(c) because the uniaxial compression *pushes* the hangingwall material away from the direction of the upper-hemisphere σ_1 point. Inversely, the τ_3 arrow is directed toward the σ_3 point, because the uniaxial tension *pulls* the hangingwall material toward the direction of upper-hemisphere σ_3 point.

Note that vectors τ_1, τ_3 and τ in Fig. 1(c) & (d) are not plotted stereographically. They are ordinary scaled representations of the vector quantities, drawn in the plane of the tracing paper, which represents the (now horizontal) plane of interest.

The magnitude given for τ_1 can be understood by imagining a right circular cylinder of material, of unit circular area, delivering a force $(\sigma_1 - \sigma_2)$ to the plane and intersecting it in an ellipse of area $1/\cos a$ (Fig. 2b). This is the force arising from the uniaxial compression. It gives rise to a stress on the plane equal to the force divided by the area of the ellipse or $(\sigma_1 - \sigma_2)\cos a$. This stress is not τ_1, however, because it is directed in the σ_1 direction, the direction of the uniaxial compression. τ_1 is its *component* acting parallel to the plane and is given by $(\sigma_1 - \sigma_2)\cos a$ times an additional $\sin a$ factor. The magnitude given for τ_3 is derived in a similar way.

The normal stress on the plane arising from the uniaxial compression is the above stress acting in the σ_1 direction $(\sigma_1 - \sigma_2)\cos a$, times an additional $\cos a$ factor which resolves it into the direction of P. This is always a compressive (positive) contribution to the total normal stress on the plane. The normal stress arising from the uniaxial tension, on the other hand, is always a tensile (negative) contribution. It is derived in the same way and is $(\sigma_3 - \sigma_2)\cos b \cos b$. The total normal stress on the plane is then the sum of these two contributions, plus the normal stress σ_2 contributed by the hydrostatic component of the stress tensor.

DISCUSSION

The construction is easiest to explain and learn using the upper-hemisphere projection. Once the principles are understood however, the usual lower-hemisphere projection may be preferred. The procedure is the same, except that one convention has to be reversed. *Either* the

τ_1 and τ_3 arrows are drawn in opposite directions (τ_1 toward σ_1, τ_3 away from σ_3) *or* the shear-sense is interpreted as footwall-relative-to-hangingwall.

The rotation steps (Figs. 1b & e) can be eliminated if desired by using the stereovectors of De Paor (1979). These permit representation and addition of τ_1 and τ_3 in their original geographic orientations. I find however that the scaling steps necessary for using stereovectors in this application are more time-consuming than the rotation steps they replace.

The unpublished method of Etchecopar (1984) is similar to the present one in two respects. The original stress state is reduced by separating out a hydrostatic component equal to one of the principal stresses, and stress vectors acting on the fault plane are summed graphically on the projection paper. However, Etchecopar subtracts a hydrostatic component equal to σ_3 instead of my σ_2, and sums vectors representing components of the *total* stress on the plane. His construction places two principal stress directions horizontal for the vector summation, where mine puts the plane horizontal. However the two methods give identical results, and Etchecopar's method could be modified slightly to yield the normal and shear stress magnitudes as well as the shear stress direction.

The three-way decomposition of a general stress state into a hydrostatic component and two orthogonal, uniaxial components is easier to understand geometrically than the usual two-way decomposition into a mean stress and a deviator. It may therefore find further applications.

Acknowledgements—I am indebted to Richard Lisle for advice, for the stimulus to my own thinking provided by his manuscript (Lisle 1989) and for pointing out to me the existence of the method of Etchecopar. I thank Arnaud Etchecopar for suggesting an improvement in the text. Supported by NSF Grant EAR 8606961.

REFERENCES

De Paor, D. G. 1979. The use of stereovectors in structural and engineering geology. *Tectonophysics* **60**, T1–T6.

Etchecopar, A. 1984. Etude des états de constraintes et tectonique cassante, et simulations de déformations plastique (approche mathématique) (Appendix IV). Unpublished thesis, University of Montpellier (USTL).

Lisle, R. J. 1989. A simple construction for shear stress. *J. Struct. Geol.* **11**, 493–495.